Finite-Size Scaling and Long-Range Interactions

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Abstract

The present review is devoted to the problems of finite-size scaling due to the presence of long-range interaction decaying at large distance as $1/r^{d+\sigma}$, where d is the spatial dimension and the long-range parameter $\sigma > 0$. Classical and quantum systems are considered.

1. Finite-size systems and critical phenomena

A common wisdom is that the singularities in the thermodynamic functions at a critical point may occur only in the thermodynamic limit. If the system is fully finite (as any real system is), no such singularities exist and, strictly speaking, no phase transitions occur. These facts naturally give rise to the following questions:

a) Why the bulk theory turns out to be adequate to the experimental evidence for finite objects?

b) How the singularities appear when no phase transitions occur in any fully finite system?

The answer to the first question lies in the fact that a macroscopic body is close, under some circumstances, to the idealized thermodynamic limit.

The answer to the second question becomes evident if one recalls that the limit function of a sequence of analytic functions needs not to be analytic. Therefore, what happens is that in the thermodynamic limit some thermodynamic functions of the system develop singularities which are attributes of phase transitions. So, in a finite system one expects an appreciable rounding of the critical point singularities.

The bulk correlation length $\xi_\infty(T)$ measures the distance over which particles in a system are significantly correlated. Then, as the temperature T approaches the critical temperature T_c, $\xi_\infty(T)$ diverges. When $\xi_\infty(T)$ attains a magnitude of the order of the characteristic size L of the finite system, then the boundary particles at the opposite sides of the system become well correlated, and ordering cannot continue to build up further in the restricted dimensions. It is certainly reasonable to expect that the rounding of the phase transition is controlled by the criterion: $\xi_\infty(T) \sim L$.

The description of this rounding and crossover was first formulated by M. Fisher (1972) and is called Finite-Size Scaling (FSS)(see e.g.[1]).

An important application of FSS is to analyze numerical data obtained from simulations on relatively small finite systems, and, thereby to obtain knowledge of the bulk systems.

Finite-size effects are much stronger in systems of particles with long-range (LR) interactions because every particle directly feels the influence of the boundaries.

To study finite systems, one must at the start address two basic issues, namely the overall geometry of the system and the specific nature of the boundary conditions.

Geometry: We will use the general notation $L^{d-d'} \times \infty^{d'}$ and distinguish three important cases (d,d' are spatial dimensions): $d' = 0$ – a fully finite system, $d' = 1$ – a system with the geometry of a cylinder, and $d' = (d-1)$ – a system with slab (or film) geometry.

Boundary conditions: Here and bellow we will consider periodic boundary conditions (p.b.c.). Close to the bulk critical point for a d-dimensional system with a finite size L one has for the free energy density

$$f_L^{(p)}(T) = f(T) + O\left(\mathrm{e}^{-L/\xi(T)}\right). \tag{1}$$

where $f(T) \equiv f_\infty(T)$ is the bulk free energy density. In this case there are no surfaces, edges and corners.

The experimental observability of FSS effects is hampered by the fact that nature does not provide us systems with p.b.c. The rapid exponential approach to the thermodynamic limit is one of the reasons why p.b.c. are preferred in Monte Carlo simulations. One observes that the l.h.s. in the above expansions is a regular function of T whereas the r.h.s. is not: the bulk density $f(T)$, for example, is singular at the critical point T_c. Therefore, Eq.(1) can hold only away from T_c. Obviously, for $T \simeq T_c$ one needs an alternative, (FSS) formulation.

2. Finite-size scaling hypothesis

Scaling hypotheses are made in the form of statements about properties of thermodynamic quantities in terms of homogeneous functions.

To simplify the further discussion we consider that: the studied system is below its upper critical dimension d_u, the infinite system has a second order phase transition at a critical temperature T_c, and the system has length L in all finite directions. The violation of these restrictions will cause complications and modifications of the standard FSS hypothesis [1].

For a physical quantity P with singular behavior in the thermodynamic limit the content of the standard FSS hypothesis is to assume the existence of a scaling function $\mathrm{P}_L(T)$ such that:

$$\mathrm{P}_L(T) \simeq \mathrm{P}_\infty(T)\mathrm{F}_L(L/\xi(T)). \tag{2}$$

As far as L is finite, $\mathrm{P}_L(T)$ must be a regular function of T. The (universal) function $\mathrm{F}_L(x)$ must compensate the singularity of $\mathrm{P}_\infty(x)$. Eq.(2) may be written in equivalent form:

$$P_L(t) \simeq L^{p_x/\nu}\mathcal{F}_L(tL^{1/\nu}), \tag{3}$$

where p_x - the critical exponent of the observable P_L, ν - the critical exponent of the correlation length and $t = (T-T_c)/T_c$. The hypothesis is expected to be valid when L is large and T approaches T_c.

What is the common status of the above hypothesis? One can usually meet the following statements, e.g. J. Cardy, in [2]: "FSS is by now well-established theoretically, at least for systems with short-range interactions ...", or quite recently Chen and Dohm in [3] : "The finite-size scaling hypothesis asserts ... in the absence of LR interaction ..."

We would like to point out the restrictions related with the nature of the interaction - one insists that it must be short-range (SR). In this situation the problem is to find out: does the FSS take place if the interaction is of the LR type? As we will explain bellow there is a positive answer to this question.

We will consider interactions which decay algebraically at large distances with the standard notation

$$u(r) \sim -1/r^{d+\sigma}, \qquad \sigma > 0. \tag{4}$$

For $\sigma \leq 0$ the interactions are nonintegrable, and so, under standard definitions, the thermodynamic limit does not exist. The case of nonintegrable interaction is beyond the scope of the present study.

The LR interaction (4) enters the expressions of the theory only through its Fourier transform [1]. The corresponding small - $\mathbf{q}$ expansion of the Fourier transform has the general form

$$v(\mathbf{q}) = v_0 + v_2\mathbf{q}^2 + v_\sigma|\mathbf{q}|^\sigma + w(\mathbf{q}) \qquad 0 < \sigma \neq 2, \tag{5}$$

with $w(\mathbf{q})/|\mathbf{q}|^\sigma \to 0$, for $\mathbf{q} \to 0$, i.e. in the long-wavelength approximation. Further on we will formally relate $\sigma = 2$, to the SR interaction since then (5) is the Fourier transform of an interaction decaying exponentially with distance.

Depending on whether in (5) σ is less or bigger than 2 we will speak about leading or subleading LR term respectively.

Nota bene: The subleading LR term does not affect the bulk critical behavior, but influences the finite-size one (for details see [4, 5, 6]).

The defined above LR interaction has to be a mimicry of the real one. Nevertheless, from the pure theoretical point of view the considered LR interaction seems to be of specific interest. The reason is that renormalization group (RG) predictions obtained on the basis of the ϵ-expansion, can be verified on ideal testing ground. The upper and lower critical dimensions are 2σ and σ respectively. Since σ is a continuous parameter the value of $\epsilon = 2\sigma - d$ or $\epsilon = d - \sigma$ would be small enough for integer values of the dimensionality. This places us in the rare situation when the outcome of the computer simulations, obtained by means of the Monte Carlo method, can be directly compared to the predictions of the ϵ-expansion with ϵ arbitrarily small.

But along this line of consideration there are some problems.

Firstly, although the general scheme of the ϵ-expansion has been widely accepted, the part of it related to LR systems has not been well understood until now (see [7] and references therein).

Secondly, the study of LR systems leads to prohibitively large computational requirements. Still recently a novel Monte Carlo algorithm which has an efficiency that is independent of the number of interactions per spin has been announced [8, 9].

Thirdly, Monte Carlo analyzes of the RG predictions typically apply FSS concepts. However, specific problems arise in the proper generalization of the FSS concepts in the case under consideration [10, 11].

3. Problems with the correct definitions of the correlation length

By definition the correlation length is

$$\xi_1(T) = -\lim_{R\to\infty}[R/\ln G_\infty(\mathbf{R};T)]. \tag{6}$$

Alternatively, one may consider the second moment of the bulk pair correlation function and define the effective correlation radius:

$$\xi_2(T) = \left[\sum_{\mathbf{R}} R^2 G_\infty(\mathbf{R};T)/G_\infty(\mathbf{R};T) \right]^{1/2}. \tag{7}$$

These two most commonly used definitions of the bulk correlation length are unambiguous and equivalent (up to a constant) in the case of exponential with the distance $R = |\mathbf{R}|$ decay of the bulk pair correlation function $G_\infty(\mathbf{R};T)$. A distinctive feature of the LR interaction is that the function $G_\infty(\mathbf{R};T)$ decays as $R^{-d+\sigma}$ when $R \to \infty$ for $T > T_c$, and both definitions yield $\xi_1(T) = \xi_2(T) = \infty$ if $\sigma < 2$.

Anyhow complications with correlations that decay as power laws in a whole domain of thermodynamic parameters, rather than only at special points, arise even in the bulk case, provoking the idea of "generic scale invariance" in the theory of phase transitions (see, e.g. [12]). The FSS under consideration encounters the same phenomena mixed with specific boundary effects. One can overcome the difficulty with the absence of good definition of the correlation length (when $\sigma < 2$) following the ideas proposed in [10, 11], where instead of $\xi(T)$ a bulk characteristic length $\lambda(T)$ is used. This characteristic length determines the length-scale of variation of the correlation function and diverges at the critical point.

A finite system has three characteristic length scales: a linear size L, a length $\lambda_L(T)$ which determines the scale of variation of the correlations, and a microscopic length a (e.g., a is the lattice spacing). The approach proposed in [11] involves the following steps:

First step: It consists of three assumptions.

The first one defines the finite-size characteristic length $\lambda_L(T)$ from the large-distance asymptotic behavior of the finite-size pair correlation function $G_L(\mathbf{R},T)$.

The second assumption is the homogeneity of any thermodynamic quantity of the finite-size system, such that its bulk limit is singular at $T = T_c$, as a function of the two dimensionless ratios $\lambda_L(T)/a$ and L/a.

The third assumption is the existence of finite thermodynamic limits for the characteristic length and the pair correlation function.

Second step: It concerns the relationship between the finite-size length $\lambda_L(T)$ and its bulk limit $\lambda_\infty(T)$; the corresponding homogeneity assumption involves the dimensionless ratios $\lambda_\infty(T)/a$ and L/a.

4. Mathematical problems

Classical case: In the case of SR interaction the following replacement is used as an indispensable part of the FSS calculations (see, e.g. [1]).

$$\sum_{\mathbf{q}} \frac{1}{m^2 + |\mathbf{q}|^2} = \int_0^\infty dt \exp(-m^2 t) \left[\sum_q \exp(-q^2 t) \right]^d, \tag{8}$$

where $\mathbf{q}$ and q are d-dimensional and one-dimensional discrete vectors, respectively. This is the so called Schwinger parametric representation. The aim of this replacement is two-fold: to reduce the d-dimensional sum to the corresponding effective one-dimensional one, and to give the dimensionality d the status of a continuous variable.

In the case of leading q^σ term in (5), one cannot use the Schwinger representation in its familiar form. Then the following generalization of (8) has been suggested [13]:

$$\sum_{\mathbf{q}} \frac{1}{m^2+|\mathbf{q}|^\sigma} = m^{\frac{4-2\sigma}{\sigma}} \int_0^\infty dt Q_\sigma(m^{4/\sigma}t) \left[\sum_q \exp(-q^2 t)\right]^d, \tag{9}$$

where the function $Q_\sigma(t)$ for $0<\sigma<2$ is given by

$$Q_\sigma(t) = \int_0^\infty dy \exp(-ty)\tilde{Q}_\sigma(y), \qquad \tilde{Q}_\sigma(y) = \frac{1}{\pi}\frac{\sin(\sigma\pi/2)y^{\sigma/2}}{1+2y^{\sigma/2}\cos(\sigma\pi/2)+y^\sigma}. \tag{10}$$

¿From (9) and (10) one obtains

$$\sum_{\mathbf{q}} \frac{1}{m^2+|\mathbf{q}|^\sigma} = \int_0^\infty dt \tilde{Q}_\sigma\left(\frac{t}{m^{\frac{4-2\sigma}{\sigma}}}\right) \sum_{\mathbf{q}} \frac{1}{tm^2+|\mathbf{q}|^2}. \tag{11}$$

The identity (11) demonstrates the possibility to reduce the LR case to the SR case with an integration over an additional parameter. So, all the mathematical machinery developed for the SR case, in principle, may be used. Let us note however that such nonlocal expressions are unsuitable for explicit calculations. It is more convenient to use the relation [14]:

$$Q_\sigma(x) = x^{\sigma/2-1}E_{\sigma/2,\sigma/2}(-x^{\sigma/2}), \tag{12}$$

where $E_{\alpha,\beta}(z)$ is an entire function of the Mittag-Leffler type defined by the power series

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)} \qquad \alpha > 0. \tag{13}$$

and to study the finite-size properties of the system resulting from the analytical properties of $E_{\alpha,\beta}$.

Pure quantum case: At $T=0$, the quantum counterpart of (9) is

$$\sum_{\mathbf{q}} \frac{1}{[m^2+|\mathbf{q}|^\sigma]^{1/2}} = \frac{2}{\pi}\int_0^\infty dp \sum_{\mathbf{q}} \frac{1}{m^2+p^2+|\mathbf{q}|^\sigma}, \tag{14}$$

Equation (14) displays the well known property of the pure quantum case. The auxiliary variable p^2 acts effectively as an extra dimension. Indeed the pure quantum system corresponds to a $d+1$ dimensional classical system with the geometry of a cylinder $L^d\times\infty^1$. Introducing $m^2(p) := m^2+p^2$ with the help of Eq.(11) from Eq.(14) one gets

$$\sum_{\mathbf{q}} \frac{1}{[m^2+|\mathbf{q}|^\sigma]^{1/2}} = \frac{2}{\pi}\int_0^\infty dt \int_0^\infty dp \tilde{Q}_\sigma\left(\frac{t}{m(p)^{\frac{4-2\sigma}{\sigma}}}\right) \sum_{\mathbf{q}} \frac{1}{tm(p)^2+|\mathbf{q}|^2}. \tag{15}$$

The integral representation (15) illustrates the mathematical difficulties appearing in the pure quantum case. It is shown that by two additional integrations the problem can be effectively reduced to the classical SR case.

On the other hand, the following modification of (9) has been prosed [15]

$$\sum_{\mathbf{q}} \frac{1}{[m^2+|\mathbf{q}|^\sigma]^{1/2}} = m^{\frac{4-\sigma}{\sigma}} \int_0^\infty dt C_\sigma(m^{4/\sigma}t) \left[\sum_q \exp(-q^2 t)\right]^d, \tag{16}$$

where

$$C_\sigma(x) = x^{\sigma/4-1} G_{\sigma/2,\sigma/4}(-x^{\sigma/2}), \tag{17}$$

i.e. the function $C_\sigma(x)$ is related to the entire function defined by the power series

$$G_{\alpha,\beta}(z) = \frac{1}{\pi^{1/2}} \sum_{k=0}^{\infty} \frac{\Gamma(k+1/2) z^k}{\Gamma(\alpha k + \beta) k!} \qquad \alpha > 0, \beta > 0. \tag{18}$$

Some analytical properties of $G_{\sigma/2,\sigma/4}(z)$ (see also [16]) may be established with the help of the identity

$$G_{\sigma/2,\sigma/4}(z) = \frac{2}{\pi} \int_0^\infty E_{\sigma/2,\sigma/2}\left(-(z+p^2)\right) dp, \tag{19}$$

obtained from Eq.(14) and the relation of its l.h.s and r.h.s. with the functions $G_{\sigma/2,\sigma/4}(z)$ and $E_{\sigma/2,\sigma/2}(z)$, respectively.

5. Theoretical models

Here we will mention the models and the related works in which leading LR interactions are considered in the context of FSS.

The mean spherical model was considered in [10, 11, 13, 14], [17]-[22]. The Hamiltonian of the model is

$$\mathcal{H} = -\frac{1}{2} \sum_{ij} \mathcal{J}_{ij}\ \mathcal{S}_i \mathcal{S}_j - H \sum_i \mathcal{S}_i, \tag{20}$$

where $\mathcal{S}_i$ is the spin variable at site i, $\mathcal{J}_{ij}$ is the interaction matrix between spins at sites i and j, and H is an ordering external magnetic field. The spins obey the spherical constraint $\sum_i \langle \mathcal{S}_i^2 \rangle = N$, where $\langle \cdots \rangle$ denotes the standard thermodynamic average taken with the Hamiltonian $\mathcal{H}$ and N is the the total number of spins.

The quantum version of model (20) was considered in [23]. The Hamiltonian of the model is

$$\mathcal{H} = \frac{g}{2} \sum_i \mathcal{P}_i^2 - \frac{1}{2} \sum_{ij} \mathcal{J}_{ij}\ \mathcal{S}_i \mathcal{S}_j + \frac{\mu}{2} \sum_i \mathcal{S}_i^2 - H \sum_i \mathcal{S}_i, \tag{21}$$

where $\mathcal{S}_i$ are spin operators at site i, the operators $\mathcal{P}_i$ are "conjugated" momenta and $[\mathcal{S}_i, \mathcal{S}_i'] = i\delta_{ii'}, [\mathcal{P}_i, \mathcal{P}_{i'}] = i\delta_{ii'}$ (with $\hbar = 1$). Model (23) in the large-n limit is equivalent to the quantum $\mathcal{O}(n)$ nonlinear sigma model.

The d-dimensional $\mathcal{O}(n)$-symmetric model was considered in [24]-[28]. The model is defined by

$$\mathcal{H}\{\varphi\} = \frac{1}{2} \int_V d^d x \left[\left(\nabla^{\sigma/2} \varphi\right)^2 + r_0 \varphi^2 + \frac{1}{2} u_0 \varphi^4 \right], \tag{22}$$

where φ is a shorthand notation for the space dependent n-component field $\varphi(x)$, $r_0 = r_{0c} + t_0$ ($t_0 \propto T - T_c$) and u_0 are model constants. The first term denotes $k^\sigma |\varphi(k)|^2$ in the momentum representation. A strongly anisotropic version of (22) with exponents depending on the direction was considered in [29].

The quantum version of model (22) is

$$\mathcal{H}\{\varphi\} = \frac{1}{2} \int_0^{1/T} d\tau \int_V d^d x \left[\left(\nabla^{\sigma/2} \varphi\right)^2 + r_0 \varphi^2 + \frac{1}{2} u_0 \varphi^4 \right], \tag{23}$$

where φ is a shorthand notation for the space-time dependent n-component field $\varphi(x, \tau)$ [16].

6. Verification of FSS

Classical phase transitions

Analytically the case $\sigma < d < 2\sigma$ has been investigated first exactly in the spherical limit ($n = \infty$) [10, 11, 13, 14], [17]-[20] and by the RG methods for $n = 1$ [25], $n \geq 1$ [26, 27]. Finite-size effects are predicted in both regions of first order [17, 18, 10, 11] and second order phase transitions [13, 19, 14, 11, 25, 26, 27]. As a result in the latter case it has been shown that the FSS relations can be written in form equivalent to Eq.(3). Some thermodynamic quantities, for instance, the shift of the critical temperature due to the finite-size effects [21], the susceptibility [20, 19], the pair correlation function [10, 18] and the Binder's cumulant at the bulk critical temperature T_c [25, 26, 27] and above it [26, 27] as a function of $\sqrt{\epsilon}$ up to $\mathcal{O}(\epsilon^{3/2})$ have been obtained.

Away from the critical region, the critical behavior of the systems is dominated by its bulk critical behavior

Nota bene: A distinctive feature of the LR case is that for $tL^{1/\nu} \gg 1$ the finite-size corrections are not exponentially small as in the SR case; they vary instead as algebraic power of the variable $tL^{1/\nu}$ [18, 10, 26].

Some problems exist with the comparison of RG analytical results and numerical data. It has been shown [25] (see also [30]) using Monte Carlo simulation (for $n = 1$), that the Binder's cumulant ratio is linear in ϵ. The analytical evaluation of the Binder's cumulant for the $\mathcal{O}(n)$ symmetric φ^4 model, however, showed that it is linear in $\sqrt{\epsilon}$. A possible way to resolve this controversy between the Monte Carlo method and the analytical results is to carry on finite-size calculations to higher loop order [25]. This could ameliorate the analytical results, which would be comparable to those obtained by numerical simulations. Here we would like to mention that higher loop corrections that are dealt with through minimal subtraction scheme (see [31, 25, 26]) and ϵ-expansion have not been performed, even for the more simple case of SR interaction and therefore much knowledge is still absent.

Quantum phase transitions

Quantum phase transitions [34] occur at zero temperature as a function of some non-thermal control parameter such as composition or pressure and are driven by quantum fluctuations. In addition to being of an interest for various low-temperature phenomena, quantum phase transitions are important because they are believed to play a crucial role in quantum information science, e.g. in the phenomenon known as entanglement - the resource that enables quantum computation and communications (see [35] and refs. therein).

The main feature of quantum systems is the coupling between statics and dynamics. As a result the dynamical critical exponent z plays an important role in scaling of the "imaginary time dimension". For nonzero temperature this extra "dimension" extends only over a finite interval L_τ. Here $L_\tau \sim 1/T$ is "finite-size" in the temporal (imaginary time) direction and so the temperature's influence in the quantum critical point my be determined by FSS effects [1].

Unlike classical models, where scaling can be done uniformly for all "spatial" dimensions, quantum models are anisotropic in general, and therefore the "space" and "imaginary-time" directions will not scale in the same fashion. According to the general

hypothesis of the FSS theory extended here to quantum (anisotropic) systems, a physical quantity $\mathcal{A}\left(r, L, T\right)$, which is singular in the thermodynamic (bulk) limit at the quantum critical point ($r = 0$), will scale like

$$\mathcal{A}(r, T, L) = b^p \mathcal{A}_s \left(r b^{1/\nu}, T b^z, b L^{-1}\right). \tag{24}$$

Here, r measures the distance from the quantum critical point. In the scaling form (24) p corresponds to the engineering dimension $d + z$ of the system in the case when the scaling function refers to the singular part of the free energy. For the other physical quantities of interest it is the critical exponent measuring the divergence of the bulk thermodynamic function $\mathcal{A}$ at the critical point divided by ν (for the correlation length $p = 1$, for the susceptibility $p = \gamma/\nu$, etc). Depending on the choice of the RG rescaling factor b we obtain different scaling functions $\mathcal{A}_s$, which are related among each other by some appropriate change of the scaling variables [16].

The fact that the inverse temperature can be used as an additional size in the imaginary-time direction creates strong anisotropy in the system and in general, one needs a shape–dependent formulation of the FSS [16, 29]. This will lead to the establishment of some changes in the scaling properties of the finite quantum system. In this case we can consider the quantum to classical and the finite-size to the bulk system [16], and as well as other [29] crossover phenomena.

Results on the Casimir effect

The correlation function serves as a measure of how fluctuations at one point are correlated with fluctuations at another point. The confinement conditions, imposed on a system with correlation function decaying as power law in space induce a LR force between the surfaces limiting the system. One can generally call this phenomenon Casimir effect. There can be different mechanisms leading to this phenomenon that are related with the scale invariance or mentioned above "generic scale invariance". Prominent examples are systems at critical points or systems with spontaneously broken global continuous symmetry that leads to massless modes: "spine waves" or Goldstone bosons (statistical-mechanical Casimir effect [32]), and the discussed for the first time by H.G.B. Casimir (1948) constrained zero-point vacuum fluctuations of the electromagnetic field (quantum - mechanical Casimir effect).

In a system with geometry $L^1 \times \infty^{d-1}$ at or bellow the critical point a LR "statistical-mechanical Casimir force" [1, 32, 33] appears. According to the standard FSS hypothesis, near the critical point of the bulk system one expects for this force (the magnetic field H=0)

$$F_{\text{Casimir}}(T, L) \simeq L^{-d} X(\kappa t L^{1/\nu}), \tag{25}$$

where $X(x)$ is an universal function and κ is a nonuniversal metric factor. In the case of SR interaction the universal scaling structure of Eq.(25) has been confirmed by RG and exact calculations [1, 33].

The investigation of systems with LR interaction possesses some peculiarities: due to the character of the interaction there exists a natural attraction between the surfaces bounding the system. In this case only results within the spherical model with p.b.c. are obtained [22]. It is possible to formulate some general statements: the Casimir force is always negative, (i.e. it is a force of attraction between the surfaces bounding the system) for any T and $\sigma \geq 1$, as well as for any $T \geq T_c$ if $\sigma < 1$; the behavior of the Casimir

force depends strongly on the range of interaction σ, being (at $T = T_c$) a monotonically increasing function of σ; in the neighborhood of $T = T_c$, the Casimir force is an increasing function of T if $\sigma > 1$, and possesses a complex behavior if $\sigma \leq 1$.

For the Casimir force there is an exact expression obtained in the framework of the quantum spherical model (see [1])

$$F_{\text{Casimir}}(T, \lambda; L) \simeq L^{-(d+z)} X_{\text{Casimir}}(x, \rho), \tag{26}$$

with scaling variable $x = L^{1/\nu}\left(\lambda^{-1} - \lambda_c^{-1}\right)$, and $\rho = L^z/L_\tau, L_\tau = \lambda/T$. Here λ is the quantum parameter that governs the transition near the quantum critical point λ_c.

The obtained exact expression for the universal scaling function $X_{\text{Casimir}}(x,\rho)$ gives the possibility of analysis including issues as the sign of the Casimir force and its monotonicity as a function of the temperature and parameter σ [23].

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